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# Globally rigid circuits of the direction–length rigidity matroid

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ABSTRACT

A two-dimensional mixed framework is a pair $(G, p)$, where $G = (V; D, L)$ is a graph whose edges are labeled as 'direction' or 'length' edges, and $p$ is a map from $V$ to $\mathbb{R}^2$. The label of an edge $uv$ represents a direction or length constraint between $p(u)$ and $p(v)$. The framework $(G, p)$ is globally rigid if every framework $(G, q)$ in which the direction or length between the end vertices of corresponding edges is the same as in $(G, p)$, can be obtained from $(G, p)$ by a translation and, possibly, a dilation by $-1$.
We characterize the globally rigid generic mixed frameworks $(G, p)$ for which the edge set of $G$ is a circuit in the associated direction–length rigidity matroid. We show that such a framework is globally rigid if and only if each 2-separation $S$ of $G$ is 'direction balanced', i.e. each 'side' of $S$ contains a direction edge. Our result is based on a new inductive construction for the family of edge-labeled graphs which satisfy these hypotheses. We also settle a related open problem posed by Servatius and Whiteley concerning the inductive construction of circuits in the direction–length rigidity matroid.



## 1. Introduction

Consider a configuration of points $p_1, p_2, \dots, p_n$ in $\mathbb{R}^d$ together with a set of constraints which fix the direction or length between some pairs $p_i, p_j$. A basic question is whether the configuration,

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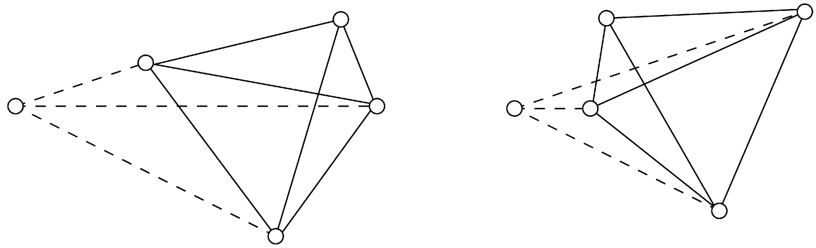

**Fig. 1.** Two equivalent but non-congruent realizations of a mixed graph in $\mathbb{R}^2$. We use solid or dashed lines to indicate edges with length or direction labels, respectively.

with the given constraints, is locally or globally unique, up to 'congruence'. Results of this type have applications in CAD [18], localization of sensor networks [5], and in determining molecular conformation [10].

We model the configuration and constraints as a 'mixed framework'. A *mixed graph* $G = (V; D, L)$ is an undirected graph together with a labeling (or bipartition) $(D, L)$ of its edge set. We refer to edges in $D$ as *direction edges* and edges in $L$ as *length edges*. A *mixed framework* is a pair $(G, p)$, where $G = (V; D, L)$ is a mixed graph and $p$ is a map from $V$ to $\mathbb{R}^d$. We say that $(G, p)$ is a *realization* of $G$ in $\mathbb{R}^d$. Two mixed frameworks $(G, p)$ and $(G, q)$ are *mixed-equivalent* (or simply *equivalent*) if (i) $p(u) - p(v)$ is a scalar multiple of $q(u) - q(v)$ for all $uv \in D$ and (ii) $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for all $uv \in L$, where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^d$. We say that $(G, p)$ is a *length framework* if $D = \emptyset$, is a *direction framework* if $L = \emptyset$, and is a *pure framework* if it is either a length or direction framework. If two pure frameworks satisfy (i) or (ii) then we say that they are direction- or length-equivalent, respectively. Note that the constraint (i) is vacuous when $d = 1$, so we may assume that $d \geqslant 2$ when we consider mixed or direction frameworks.

The mixed frameworks $(G, p)$ and $(G, q)$ are *mixed-congruent* (or simply *congruent*) if (i) $p(u) - p(v)$ is a scalar multiple of $q(u) - q(v)$ and (ii) $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for all $u, v \in V$. We can define direction-congruence and length-congruence in a similar way for pure frameworks by imposing only (i) or (ii) above. See Fig. 1 for an example of two equivalent but non-congruent mixed frameworks.

Note that two mixed frameworks are congruent if and only if one can be obtained from the other by translations and dilations by $-1$. Similarly, two pure frameworks are direction-congruent (length-congruent) if and only if one can be obtained from the other by translations and arbitrary dilations (respectively, combinations of translations and rotations, and reflections).

The mixed framework $(G, p)$ is *globally mixed-rigid* in $\mathbb{R}^d$ if every framework which is equivalent to $(G, p)$ is congruent to $(G, p)$. *Global direction-rigidity* and *global length-rigidity* of pure frameworks are defined analogously. It is a hard problem to decide if a given length framework is globally length-rigid. Indeed Saxe [17] has shown that this problem is NP-hard even for 1-dimensional length frameworks. This implies that it is also NP-hard to decide if a mixed framework is globally mixed-rigid in $\mathbb{R}^d$, for all $d \geqslant 1$. To see this consider a mixed graph $G = (V; D, L)$ in which all pairs of vertices are joined by a direction edge and let $(G, p)$ be a realization of $G$ in $\mathbb{R}^d$ in which all points $p(v)$ lie on a line. Then $(G, p)$ is globally mixed-rigid in $\mathbb{R}^d$ if and only if the corresponding realization of the length-pure subgraph $H = (V, L)$ of $G$ as a 1-dimensional length framework is globally length-rigid. On the other hand, Whiteley [19] showed that the global direction-rigidity of $d$-dimensional direction frameworks is polynomially decidable for all $d$.

The problem of deciding if a given length framework $(G, p)$ is globally length-rigid becomes more tractable if we assume that there are no algebraic dependencies between the coordinates of the points $p(v)$. A framework $(G, p)$ is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over the rationals. A recent result of Gortler et al. [7] implies that the global length-rigidity of $d$-dimensional length frameworks is a *generic property* in the sense that the global rigidity of a $d$-dimensional generic length framework $(G, p)$ depends only on the graph $G$. The result of [7] does not provide a 'good characterization' (or a deterministic algorithm) for determining those graphs which give rise to globally rigid generic frameworks in $\mathbb{R}^d$. Such characterizations have been found for $d = 1, 2$, however, as we shall see below.

Whiteley's results [19] imply that the global direction-rigidity of $d$-dimensional direction frameworks $(G, p)$ is also a generic property. Furthermore, the characterization of globally direction-rigid graphs is known for all $d$ [19]. It is not known whether the global mixed-rigidity of $d$-dimensional mixed frameworks is a generic property, even in the special case when $d = 2$. The main results of this paper will imply that global mixed-rigidity is a generic property for certain families of mixed frameworks.

A closely related concept to global rigidity, which plays a key role in the understanding of global rigidity of pure frameworks, is that of rigidity. A mixed framework $(G, p)$ is *mixed-rigid* if there exists an $\epsilon > 0$ such that every mixed framework $(G, q)$ which is equivalent to $(G, p)$ and satisfies $\|p(v) - q(v)\| \leqslant \epsilon$ for all $v \in V$, is congruent to $(G, p)$. *Direction*- and *length-rigidity* of pure frameworks are defined analogously. Mixed-, direction-, and length-rigidity are all generic properties of their respective types of frameworks.

Our main concern in this paper is 2-dimensional mixed frameworks and we will assume henceforth that all mixed frameworks are 2-dimensional unless specified otherwise. One can develop a rigidity theory for 2-dimensional mixed frameworks in much the same way as for pure frameworks. For $(x, y) \in \mathbb{R}^2$ let $(x, y)^\perp = (y, -x)$. The *direction–length rigidity matrix* of a mixed framework $(G, p)$ is the matrix $R(G, p)$ of size $(|D| + |L|) \times 2|V|$, where, for each edge $uv \in D \cup L$, in the row corresponding to $uv$, the entries in the two columns corresponding to the vertex $w$ are given by: $(p(u) - p(v))^\perp$ if $uv \in D$ and $w = u$; $-(p(u) - p(v))^\perp$ if $uv \in D$ and $w = v$; $(p(u) - p(v))$ if $uv \in L$ and $w = u$; $-(p(u) - p(v))$ if $uv \in L$ and $w = v$; $(0, 0)$ if $w \notin \{u, v\}$. The rigidity matrix of $(G, p)$ defines the *direction–length rigidity matroid* of $(G, p)$ on the ground set $D \cup L$ by linear independence of the rows of the rigidity matrix. The maximum possible rank of $R(G, p)$ is $2|V| - 2$. The framework $(G, p)$ is said to be *mixed-independent* if the rows of $R(G, p)$ are linearly independent, and *infinitesimally mixed-rigid* if $\operatorname{rank} R(G, p) = 2|V| - 2$. Any two generic realizations of $G$ will have the same rigidity matroid. We call this the *direction–length rigidity matroid* $\mathcal{R}(G) = (D \cup L, r)$ of $G$. The fact that all generic realizations of $G$ have the same rigidity matroid implies that mixed-independence and infinitesimal mixed-rigidity are both generic properties.

The following lemma relates infinitesimal mixed-rigidity to mixed-rigidity.

**Lemma 1.1.** *(See [13].) Let $(G, p)$ be a mixed framework. If $(G, p)$ is infinitesimally mixed-rigid then $(G, p)$ is mixed-rigid. Furthermore, if $(G, p)$ is generic, then $(G, p)$ is mixed-rigid if and only if $(G, p)$ is infinitesimally mixed-rigid.*

Lemma 1.1 implies that mixed-rigidity is a generic property since it is equivalent to infinitesimal mixed rigidity for generic frameworks.

We denote the rank of the rigidity matroid $\mathcal{R}(G)$ of a mixed graph $G$ by $r(G)$ and say that $G$ is *mixed-independent*, or *mixed-rigid*, if $r(G) = |D| + |L|$, or $r(G) = 2|V| - 2$, respectively. Thus, if $(G, p)$ is a generic realization of $G$, then the mixed framework $(G, p)$ is mixed-rigid if and only if the mixed graph $G$ is mixed-rigid. It will be convenient to use a similar terminology for global mixed-rigidity. We say that the mixed graph $G$ is *globally mixed-rigid* if every generic realization of $G$ is globally mixed-rigid. Note that since global mixed-rigidity is not known to be a generic property, it is conceivable that a graph which is not globally mixed-rigid could still have a globally mixed-rigid generic realization.

*Direction* and *length rigidity matrices* and *matroids* can be defined similarly for pure frameworks. These in turn give rise to direction and length versions of *independence* and *infinitesimal rigidity* for pure frameworks, and to direction and length versions of *independence* and *rigidity* for (unlabeled) graphs. Henceforth, we will suppress the prefixes mixed, direction, and length when they are clear from the context.

Length frameworks correspond to the well-studied bar-and-joint frameworks, for which characterizations of generic rigidity and generic global rigidity are known up to dimension two. (We refer the reader to [8,19] for a detailed survey of the rigidity of $d$-dimensional length frameworks.) A graph is length-rigid in $\mathbb{R}$ if and only if it is connected. The characterization of length-rigid graphs in $\mathbb{R}^2$ is based on the following characterization of length-independent graphs due to Laman. For $G = (V, E)$

a graph and $X \subseteq V$, let $E(X)$ denote the set, and $i(X)$ the number, of edges in $G[X]$, that is, in the subgraph induced by $X$ in $G$.

**Theorem 1.2.** *(See [15].) A graph $G = (V, E)$ is length-independent in $\mathbb{R}^2$ if and only if $i(X) \leqslant 2|X| - 3$ for all $X \subseteq V$ with $|X| \geqslant 2$.*

Laman's theorem was used to give a characterization of graphs which are length-rigid in $\mathbb{R}^2$ by Lovász and Yemini [16].

We need some more terminology to describe the characterizations of graphs which are globally length-rigid in $\mathbb{R}^d$ for $d \leqslant 2$. A graph $G = (V, E)$ is *k-connected* if $|V| \geqslant k + 1$ and $G - X$ is connected for all $X \subset V$ with $|X| \leqslant k - 1$. It is *redundantly length-rigid* in $\mathbb{R}^d$ if $G - e$ is length-rigid in $\mathbb{R}^d$ for all edges $e$ of $G$. Hendrickson [9] showed that being $(d + 1)$-connected and redundantly length-rigid in $\mathbb{R}^d$ are necessary conditions for $G$ to be globally length-rigid in $\mathbb{R}^d$ and conjectured that these conditions are also sufficient. His conjecture is true when $d \leqslant 2$. It is not difficult to show that a graph $G$ is globally length-rigid in $\mathbb{R}$ if and only if either $G$ is the complete graph on two vertices or $G$ is 2-connected. The characterization in $\mathbb{R}^2$ is more difficult.

**Theorem 1.3.** *(See [4,12].) A graph $G$ is globally length-rigid in $\mathbb{R}^2$ if and only if either $G$ is a complete graph on two or three vertices, or $G$ is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

Connelly [3] showed that Hendrickson's conjecture is false when $d \geqslant 3$. As noted earlier, a characterization of globally length-rigid graphs in $\mathbb{R}^d$, $d \geqslant 3$, in terms of efficiently checkable conditions, is not yet known.

The linearity of the direction constraints in a $d$-dimensional direction framework $(G, p)$ implies that direction-rigidity, infinitesimal direction-rigidity and global direction-rigidity are all equivalent for direction frameworks. Thus we can determine whether a $d$-dimensional direction framework is (globally) rigid by calculating the rank of a 'direction rigidity matrix'. Whiteley [19] characterized graphs which are (globally) direction-rigid in $\mathbb{R}^d$. In the 2-dimensional case, there is a simple transformation which shows that a graph $G$ is direction-independent, or (globally) direction-rigid, if and only if it is length-independent, or length-rigid, respectively. In particular, Theorem 1.2 gives

**Theorem 1.4.** *(See [19].) A graph $G = (V, E)$ is direction-independent in $\mathbb{R}^2$ if and only if $i(X) \leqslant 2|X| - 3$ for all $X \subseteq V$ with $|X| \geqslant 2$.*

Similarly the above mentioned characterization of length-rigid graphs due to Lovász and Yemini yields a characterization of graphs which are (globally) direction-rigid in $\mathbb{R}^2$.

Mixed graphs which are independent (in $\mathbb{R}^2$) were characterized by Servatius and Whiteley. For $G = (V; D, L)$ a mixed graph and $X \subseteq V$, let $E_D(X)$ and $E_L(X)$ denote the sets, and $i_D(X)$ and $i_L(X)$ the numbers, of direction and length edges, respectively, in $G[X]$.

**Theorem 1.5.** *(See [18].) A mixed graph $G = (V; D, L)$ is independent if and only if, for all $X \subseteq V$ with $|X| \geqslant 2$,*

$$i(X) \leqslant 2|X| - 2, \tag{1}$$

*and*

$$i_D(X) \leqslant 2|X| - 3 \quad \textit{and} \quad i_L(X) \leqslant 2|X| - 3. \tag{2}$$

It is straightforward to use this result to obtain a characterization of rigid mixed graphs. The problem of characterizing when a mixed graph is globally rigid remains open, however. We give a characterization for globally rigid mixed graphs in which the edge set is a circuit in the direction–length rigidity matroid. This complements the results on globally length-rigid graphs whose edge set is a circuit in the length-rigidity matroid [1], and may serve as a building block to a complete characterization.

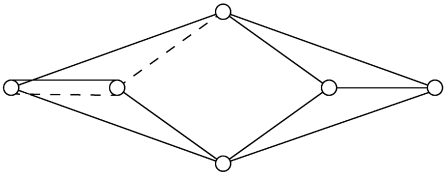

**Fig. 2.** A mixed graph with a direction unbalanced 2-separation.

### *1.1. Main results*

We first give necessary conditions for the global rigidity of a generic mixed framework. We need the following concepts. Let $G = (V; D, L)$ be a mixed graph. We say that $G$ is *k-connected* if and only if its underlying unlabeled graph is $k$-connected. A *k-vertex-cut* (*k-edge-cut*) of $G$ is a set of $k$ vertices (edges) whose removal disconnects $G$. We say that $G$ is *k-edge-connected* if all its edge-cuts have size at least $k$. A *k-separation* of $G$ is a pair of subgraphs $G_1, G_2$ such that $G = G_1 \cup G_2$, $|V(G_1) \cap V(G_2)| = k$ and $V(G_1) - V(G_2) \neq \emptyset \neq V(G_2) - V(G_1)$. We say that a 2-separation $(G_1, G_2)$ of $G$ is *direction-balanced*, respectively *length-balanced*, if both $G_1$ and $G_2$ contain an edge in $D$, respectively $L$, and is *balanced* if it is both direction-balanced and length-balanced. A 2-separation which is not (direction-, length-) balanced is said to be (*direction-*, *length-*) *unbalanced*. A mixed graph is (*direction-*, *length-*) *balanced* if all its 2-separations are (direction-, length-) balanced. An example of a graph which is not direction balanced is given in Fig. 2.

**Lemma 1.6.** *Let $(G, p)$ be a generic realization of a mixed graph $G = (V; D, L)$ with at least three vertices. Suppose that $(G, p)$ is globally rigid. Then*

(a) *$G$ is rigid,*
(b) *$G$ is 2-connected,*
(c) *$G$ is direction balanced,*
(d) *the only 2-edge-cuts which can occur in $G$ consist of two direction edges incident with a common vertex of degree two.*

**Proof.** (a) This follows from the definitions of mixed-rigidity and global rigidity and Lemma 1.1.

(b) Suppose that $G - v$ is disconnected for some vertex $v$ and let $H$ be a component of $G - v$. Applying a dilation by $-1$ centred on $p(v)$ to the points $p(x)$, $x \in V(H)$, gives a realization of $G$ which is equivalent but not congruent to $(G, p)$.

(c) Let $(H_1, H_2)$ be a direction-unbalanced 2-separation of $G$, where $H_2$ is length pure and $V(H_1) \cap V(H_2) = \{u, v\}$. Let $(G, q)$ be the realization of $G$ obtained by reflecting $p(x)$ in the line through $p(u)$, $p(v)$ for each $x \in V(H_2)$. Then $(G, q)$ is equivalent to $(G, p)$ but $\|p(x) - p(y)\| \neq \|q(x) - q(y)\|$ for all $x \in V(H_2) - \{u, v\}$, $y \in V(H_1) - \{u, v\}$. Thus $(G, p)$ is not globally rigid.

(d) Suppose that $G - \{e, f\}$ has two connected components $H_1, H_2$. Let $e = uv$ and $f = wt$, where $u, w \in V(H_1)$. We first consider the case when $e, f \in D$ and $H_1, H_2$ both have at least two vertices. Let $Q$ be the point of intersection of the lines through $p(u), p(v)$ and $p(w), p(t)$, respectively. Since $(G, p)$ is generic, $Q$ exists. Applying a dilation by $-1$ with centre $Q$ to $p(x)$, $x \in V(H_2)$, yields a realization of $G$ which is equivalent but not congruent to $(G, p)$.

We next consider the case when $e \in D$ and $f \in L$. Since $(G, p)$ is generic, the line through $p(w)$ with slope $p(u) - p(v)$ and the circle centred on $p(t)$ with radius $\|p(t) - p(w)\|$ intersect at $p(w)$ and at another point $S$. Let $(G, q)$ be the realization of $G$ obtained by translating $p(x)$ by $S - p(w)$ for each $x \in V(H_1)$. Then $(G, q)$ is equivalent but not congruent to $(G, p)$.

Finally we consider the case when $e, f \in L$. Let $Q$ be the point in $\mathbb{R}^2$ with position vector $p(w) - p(u) + p(v)$. Let $C_1$ be the circle centred at $Q$ with radius $\|p(u) - p(v)\|$ and $C_2$ be the circle centred at $p(t)$ with radius $\|p(w) - p(t)\|$. Then $C_1$ and $C_2$ intersect at $p(w)$. Since $(G, p)$ is generic they have a second point of intersection $R$. Let $(G, q)$ be the realization of $G$ obtained by translating

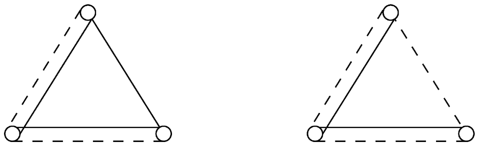

**Fig. 3.** The two mixed circuits on three vertices. These graphs, denoted by $K_3^+$ and $K_3^-$, are the smallest (mixed) circuits of the direction–length rigidity matroid.

$p(x)$ by $R - p(w)$ for each $x \in V(H_1)$. Then $(G, q)$ is equivalent but not congruent to $(G, p)$. This proves (d). □

Note that conditions (a)–(d) of Lemma 1.6 are necessary conditions for a mixed graph $G$ to be globally rigid. They are not sufficient since there exist mixed graphs which satisfy these conditions and are not globally rigid, see for example the graph in Fig. 1.

Lemma 1.6(a) implies that mixed-rigidity is a necessary condition for global mixed-rigidity. Unlike in generic length frameworks, however, redundant mixed-rigidity is not a necessary condition for global mixed-rigidity. We may use Theorem 1.5 and the fact that direction-rigidity is equivalent to global direction-rigidity for direction frameworks to deduce that a rigid generic mixed framework with exactly $2|V| - 3$ direction edges and one length edge, is globally mixed-rigid. Such a mixed framework is clearly not redundantly mixed-rigid.

We next describe some sufficient conditions for global mixed-rigidity. We use the following operations. A 0-*extension* of a mixed graph $G = (V; D, L)$ adds a new vertex $v$ and new edges $vu, vw$ for vertices $u, w \in V$ with the proviso that, if $u = w$, then the two edges from $v$ to $u$ are of different type. A 1-*extension* (on edge $uw$ and vertex $z$) for $G$ deletes an edge $uw$ and adds a new vertex $v$ and new edges $vu, vw, vz$ for some vertex $z \in V$, with the provisos that at least one of the new edges has the same type as the deleted edge and, if $z = u$, then the two edges from $v$ to $u$ are of different type. We showed in [13] that 1-extension preserves global rigidity in redundantly rigid generic mixed frameworks. (See [14] for a similar result concerning length frameworks.)

**Theorem 1.7.** *(See [13].) Let $G$ and $H$ be mixed graphs with $|V(H)| \geqslant 3$. Suppose that $G$ can be obtained from $H$ by a 1-extension on an edge $uw$. Suppose further that $H$ is globally mixed-rigid and $H - uw$ is mixed-rigid. Then $G$ is globally mixed-rigid.*

We also showed that a special kind of 0-extension preserves global rigidity.

**Theorem 1.8.** *(See [13].) Let $G$ and $H$ be mixed graphs with $|V(H)| \geqslant 2$. Suppose that $G$ can be obtained from $H$ by a 0-extension which adds a vertex $v$ incident to two direction edges. Then $G$ is globally mixed-rigid if and only if $H$ is globally mixed-rigid.*

Note that if $G$ is obtained from $H$ by a 0-extension then $G$ cannot be redundantly mixed-rigid.

We will use Theorems 1.7 and 1.8 to show that a special family of mixed graphs are globally rigid. A mixed graph $G = (V; D, L)$ is a *circuit* if $D \cup L$ is a circuit in the direction–length rigidity matroid. A circuit is a *mixed circuit* if it contains both direction and length edges, a *direction* (*length*) circuit if it contains only direction (length) edges, and a *pure circuit* if it is either a direction circuit or a length circuit. The two smallest mixed circuits are shown in Fig. 3. Theorem 1.5 implies that mixed circuits are redundantly rigid mixed graphs with $|D| + |L| = 2|V| - 1$, see Lemma 3.1 below.

We will need one further operation on mixed graphs. Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs with $V_1 \cap V_2 = \{u, v\}$ and $E_1 \cap E_2 = \{uv\}$. Then we say that the graph $G = (G_1 - uv) \cup (G_2 - uv)$ is a 2-*sum of* $G_1$ *and* $G_2$, and write $G = G_1 \oplus_2 G_2$. When $G_i = (V_i; D_i, L_i)$ is a mixed graph for each $i \in \{1, 2\}$ and $uv$ has the same type in both $G_1$ and $G_2$, their 2-sum is the mixed graph $(V_1 \cup V_2; (D_1 \cup D_2) - \{uv\}, (L_1 \cup L_2) - \{uv\})$. The mixed graph in Fig. 2 is an example of a 2-sum.

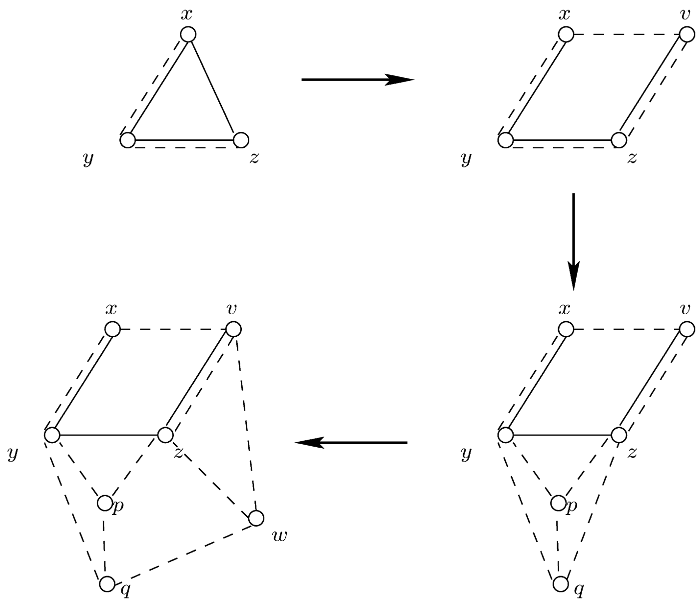


**Fig. 4.** The inductive construction of a direction balanced (and hence globally rigid) mixed circuit. The graph is obtained from $K_3^+$ by a 1-extension (adding vertex $v$), followed by a 2-sum with a direction-pure $K_4$ (on vertex set $\{y, z, p, q\}$) and then another 1-extension which adds $w$.

Our main results are the following:

- A mixed graph is a mixed circuit if and only if it can be obtained from $K_3^+$ or $K_3^-$ by a sequence of 1-extensions and 2-sums with pure $K_4$'s (see Theorem 4.12 below). This solves an open problem raised by Servatius and Whiteley in [18].
- A mixed graph is a direction balanced mixed circuit if and only if it can be obtained from $K_3^+$ or $K_3^-$ by a sequence of 1-extensions and 2-sums with direction-pure $K_4$'s (see Theorem 5.10 below). The construction is illustrated in Fig. 4.
- A generic realization of a mixed circuit $G$ is globally rigid if and only if $G$ is direction balanced (see Theorem 6.2 below).

The organization of the paper is as follows. In Section 2 we prove a number of preliminary lemmas on the structure of independent mixed graphs. Mixed circuits are introduced in Section 3. The inductive constructions for mixed circuits and direction balanced mixed circuits are obtained in Sections 4 and 5, respectively. The characterization of globally rigid mixed circuits is deduced in Section 6, while Section 7 contains additional remarks on algorithmic aspects and possible extensions.

## 2. Independent graphs and critical sets

Let $G = (V; D, L)$ be a mixed graph and $X, Y, Z \subset V$. We use $d(X, Y)$ to denote the number of edges of $G$ joining $X - Y$ to $Y - X$, and put $d(X) = d(X, V - X)$. When $X = \{x\}$ we abbreviate $d(X)$ to $d(x)$ and refer to $d(x)$ as the *degree* of $x$. We use $d(X, Y, Z)$ to denote the number of edges of $G$ which belong to $G[X \cup Y \cup Z]$ but not to $G[X] \cup G[Y] \cup G[Z]$.

We shall need the following equalities, which are easy to check by counting the contribution of an edge to each of their two sides.

**Lemma 2.1.** *Let $G$ be a graph and $X, Y \subseteq V(G)$. Then*

$$i(X) + i(Y) + d(X, Y) = i(X \cup Y) + i(X \cap Y). \tag{3}$$

**Lemma 2.2.** *Let $G$ be a graph and $X, Y, Z \subseteq V(G)$. Then*

$$i(X) + i(Y) + i(Z) + d(X, Y, Z) = i(X \cup Y \cup Z) + i(X \cap Y) + i(X \cap Z) + i(Y \cap Z) - i(X \cap Y \cap Z).$$

Let $G = (V; D, L)$ be a mixed graph and $X \subseteq V$ with $|X| \geqslant 2$ and $G[X]$ independent. Then $X$ is *mixed critical* if $i(X) = 2|X| - 2$, *direction critical* if $i_D(X) = 2|X| - 3$ and $E_L(X) = \emptyset$, and *length critical* if $i_L(X) = 2|X| - 3$ and $E_D(X) = \emptyset$. We say that $X$ is *pure critical* if $X$ is either direction critical or length critical, and $X$ is *critical* if $X$ is either mixed critical or pure critical.

The following lemma summarizes the connectivity properties of subgraphs induced by critical sets.

**Lemma 2.3.** *Let $G = (V; D, L)$ be a mixed graph and let $X \subseteq V$ be a critical set. Put $H = G[X]$. Then*

(a) *$H$ is 2-edge-connected unless $X$ is a pure critical set, $|X| = 2$, and $i(X) = 1$.*
(b) *If $(J_1, J_2)$ is a 1-separation in $H$ then $X$ is mixed critical and $V(J_1)$, $V(J_2)$ are also mixed critical.*

**Proof.** Suppose that $H$ can be disconnected by deleting less than two edges. Then there is a proper subset $A$ of $X$ with $d_H(A) \leqslant 1$. Hence

$$2|X| - 3 \leqslant i(X) \leqslant i(A) + i(X - A) + 1 \leqslant 2|A| - 2 + 2|X - A| - 2 + 1 = 2|X| - 3.$$

Thus equality must hold everywhere, which implies that $X$ is pure critical and $|A| = 1 = |X - A|$. This proves (a).

Now consider a 1-separation in $H$ and let $V_i = V(J_i)$, $i = 1, 2$. Suppose that $X$ is pure critical. Then

$$2|X| - 3 = i(X) = i(V_1) + i(V_2) \leqslant 2|V_1| - 3 + 2|V_2| - 3 = 2|V| - 4,$$

a contradiction. Thus $X$ is mixed critical. The previous inequality, when applied to a mixed critical set $X$, gives that $V_i$ is also mixed critical for $i = 1, 2$. This proves (b). □

We next consider the properties of intersecting critical sets in an independent mixed graph.

**Lemma 2.4.** *Let $G = (V; D, L)$ be an independent mixed graph.*

(a) *If $X, Y$ are mixed critical sets with $X \cap Y \neq \emptyset$ then $X \cap Y$ and $X \cup Y$ are both mixed-critical and $d(X, Y) = 0$.*
(b) *If $X, Y$ are direction (respectively length) critical sets with $|X \cap Y| \geqslant 2$ then either*
   (i) *$d(X, Y) = 0$ and $X \cap Y$ and $X \cup Y$ are both direction (respectively length) critical, or*
   (ii) *$d(X, Y) = 1$, $X \cup Y$ is mixed critical, and $i_D(X \cup Y) = 2|X \cup Y| - 3$ (respectively $i_L(X \cup Y) = 2|X \cup Y| - 3$) holds.*
(c) *If $X$ is mixed critical and $Y$ is pure critical with $|X \cap Y| \geqslant 2$ then $X \cup Y$ is mixed critical, $X \cap Y$ is pure critical and $d(X, Y) = 0$.*
(d) *If $X$ is length critical and $Y$ is direction critical with $|X \cap Y| \geqslant 2$ then $X \cup Y$ is mixed critical, $d(X, Y) = 0$, and $|X \cap Y| = 2$.*

**Proof.** The lemma follows easily from Theorem 1.5 and Lemma 2.1. For example, we may verify (d) as follows. Since $X$ is length pure and $Y$ is direction pure we have $i(X \cap Y) = 0$. Hence

$$\begin{aligned} 2|X| - 3 + 2|Y| - 3 &= i(X) + i(Y) \\ &= i(X \cap Y) + i(X \cup Y) - d(X, Y) \\ &\leqslant 2|X \cup Y| - 2 - d(X, Y) \\ &= 2|X| + 2|Y| - 2|X \cap Y| - 2 - d(X, Y). \end{aligned}$$

Thus $d(X, Y) = 0$, $|X \cap Y| = 2$, and $X \cup Y$ is mixed critical. □

**Lemma 2.5.** *Let $G = (V; D, L)$ be an independent mixed graph and let $X, Y, Z$ be critical sets satisfying $|X \cap Y| = |Y \cap Z| = |Z \cap X| = 1$ and $X \cap Y \cap Z = \emptyset$.*

(a) *If $X$ is mixed critical then $Y, Z$ are both pure critical, $X \cup Y \cup Z$ is mixed critical, and $d(X, Y, Z) = 0$.*
(b) *If $X, Y, Z$ are direction (respectively length) critical then either*
  (i) *$d(X, Y, Z) = 0$ and $X \cup Y \cup Z$ is direction (respectively length) critical, or*
  (ii) *$d(X, Y, Z) = 1$, $X \cup Y \cup Z$ is mixed critical, and $i_D(X \cup Y \cup Z) = 2|X \cup Y \cup Z| - 3$ (respectively $i_L(X \cup Y \cup Z) = 2|X \cup Y \cup Z| - 3$) holds.*

**Proof.** (a) Since $G$ is independent and the sets $X, Y, Z$ are critical, Theorem 1.5 and Lemma 2.2 imply that

$$\begin{aligned}
2|X| - 2 + 2|Y| - 3 + 2|Z| - 3 &\leqslant i(X) + i(Y) + i(Z) \\
&= i(X \cup Y \cup Z) - d(X, Y, Z) \\
&\leqslant 2\big(|X \cup Y \cup Z|\big) - 2 - d(X, Y, Z) \\
&= 2\big(|X| + |Y| + |Z| - 3\big) - 2 - d(X, Y, Z) \\
&= 2|X| - 2 + 2|Y| - 3 + 2|Z| - 3 - d(X, Y, Z).
\end{aligned}$$

Hence $d(X, Y, Z) = 0$, $X \cup Y \cup Z$ is mixed critical, and $Y, Z$ are both pure critical.

The proof of (b) is similar. □

## 3. Circuits in the direction–length rigidity matroid

We can use Theorem 1.5 to determine when a mixed graph is a circuit.

**Lemma 3.1.** *A mixed graph $G = (V; D, L)$ is a mixed circuit if and only if*

(a) *$|D| + |L| = 2|V| - 1$,*
(b) *$i(X) \leqslant 2|X| - 2$ for all $X \subset V$ with $2 \leqslant |X| \leqslant |V| - 1$, and*
(c) *$i_D(X) \leqslant 2|X| - 3$ and $i_L(X) \leqslant 2|X| - 3$ for all $X \subseteq V$ with $|X| \geqslant 2$.*

**Lemma 3.2.** *A mixed graph $G = (V; D, L)$ is a pure circuit if and only if*

(a) *$|D| + |L| = 2|V| - 2$ and either $D = \emptyset$ or $L = \emptyset$, and*
(b) *$i(X) \leqslant 2|X| - 3$ for all $X \subseteq V$ with $2 \leqslant |X| \leqslant |V| - 1$.*

It follows that, if $G$ is a circuit, then the graph $\tilde{G}$ obtained from $G$ by interchanging the direction and length edges is also a circuit. In addition, if $G$ is a mixed circuit then $|D| \geqslant 2$ and $|L| \geqslant 2$. The smallest mixed circuits, denoted by $K_3^+$ and $K_3^-$, are obtained from a cycle on three length (respectively direction) edges by adding two non-parallel direction (respectively length) edges, see Fig. 3.

**Lemma 3.3.** *Let $G = (V; D, L)$ be a circuit. Then $G$ is 3-edge-connected and 2-connected.*

**Proof.** Suppose $G$ is a mixed circuit. Let $X$ be a proper subset of $V$ and put $Y = V - X$. We have $|D \cup L| = i(X) + i(Y) + d(X) \leqslant 2|X| - 2 + 2|Y| - 2 + d(X) = 2|V| - 4 + d(X) = |E| - 3 + d(X)$. This implies $d(X) \geqslant 3$ and hence $G$ is 3-edge-connected. Similar arguments can be used to show that pure circuits are 3-edge connected and all circuits are 2-connected. (The case when $G$ is pure also follows from [1, Lemma 2.4].) □

Let $V_3 = \{v \in V\colon\ d(v) = 3\}$ denote the set of vertices of degree three in a mixed graph $G = (V; D, L)$. For convenience, vertices of degree three will be called *nodes*. We call $G[V_3]$ the *node* *subgraph of* $G$. A node of $G$ with degree at most one (exactly two, exactly three) in $G[V_3]$ is called a *leaf node* (*series node*, *branching node*, respectively). A node $v$ is *pure* if all edges incident with $v$ are of the same type. Otherwise $v$ is *mixed*.

**Lemma 3.4.** *Let $G = (V; D, L)$ be a mixed circuit. Then $G[V_3]$ is a forest.*

**Proof.** Suppose that $C$ is a cycle in the node subgraph of $G$. If $V(C) = V(G)$ then each vertex of $G$ is a node. Thus $4|V| - 2 = 2|D \cup L| = 3|V|$, which implies $|V| = 2$ and $|E| = 3$, a contradiction. (Since each circuit on two vertices is pure and has two edges.) So we may assume that $X = V - V(C) \neq \emptyset$. Since each vertex of $C$ is a node of $G$ we have $i(V(C)) + d(V(C)) \leqslant 2|V(C)|$. Thus $i(X) = 2|V| - 1 - i(V(C)) - d(V(C)) \geqslant 2|V| - 1 - 2|V(C)| = 2(|V| - |V(C)|) - 1 = 2|X| - 1$. This contradicts the fact that $G$ is a mixed circuit. $\square$

**Lemma 3.5.** *Let $G = (V; D, L)$ be a mixed circuit and let $X \subset V$ be a mixed critical set. Then there is a node of $G$ in $V - X$.*

**Proof.** Let $Y = V - X$. Since $G$ is 3-edge-connected, we have $d(Y) \geqslant 3$. Since $i(Y) + d(Y) = |D \cup L| - i(X) = 2|V| - 1 - 2|X| + 2 = 2|Y| + 1$, we obtain

$$\sum_{v \in Y} d(v) = 2i(Y) + d(Y) = 4|Y| + 2 - d(Y) \leqslant 4|Y| - 1.$$

This implies the lemma. $\square$

(Pure versions of Lemmas 3.4 and 3.5 are given in [1].)

It is straightforward to use Lemmas 3.1 and 3.2 to deduce the following results on 1-extensions and 2-sums of mixed circuits.

**Lemma 3.6.** *Let $G$ be a mixed circuit and $H$ be a 1-extension of $G$. Then $H$ is a mixed circuit.*

**Lemma 3.7.** *Let $G$ be a mixed graph.*

(a) *Suppose $G$ is the 2-sum of two mixed graphs $G_1$ and $G_2$. If $G_1$ is a mixed circuit and $G_2$ is a pure circuit, then $G$ is a mixed circuit.*
(b) *Suppose $G$ is a mixed circuit and $(H_1, H_2)$ is a 2-separation of $G$, where $V(H_1) \cap V(H_2) = \{u, v\}$ and $H_2$ is pure. Let $G_i$ be obtained from $H_i$ by adding a new edge $uv$ of the same type as the edges of $H_2$. Then $G_1$ is a mixed circuit, $G_2$ is a pure circuit, $G = G_1 \oplus_2 G_2$, $d_G(u) \geqslant 4$ and $d_G(v) \geqslant 4$.*

The mixed graph in Fig. 2 is an example of a 2-sum of a mixed circuit and a length-pure circuit.

Our final result of this section restricts the ways in which two 2-separations in a mixed circuit can ‘cross’. Let $(H_1, H_2)$ and $(H_1', H_2')$ be two 2-separations in a mixed graph $G$ with $V(H_1) \cap V(H_2) = \{u, v\}$ and $V(H_1') \cap V(H_2') = \{u', v'\}$. We say that $(H_1, H_2)$ and $(H_1', H_2')$ *cross* if both $V(H_1') - V(H_2')$ and $V(H_2') - V(H_1')$ intersect $\{u, v\}$. Note that this relation is symmetric since it implies that both $V(H_1) - V(H_2)$ and $V(H_2) - V(H_1)$ intersect $\{u', v'\}$. The concept is illustrated schematically in Fig. 5.

**Lemma 3.8.** *Let $G$ be a mixed circuit and $(H_1, H_2)$, $(H_1', H_2')$ be two 2-separations of $G$. Suppose that $H_2$ is pure. Then $(H_1, H_2)$ and $(H_1', H_2')$ do not cross.*

**Proof.** Suppose the lemma is false. Let $X_1 = V(H_1)$, $X_2 = V(H_2) \cap V(H_1')$ and $X_3 = V(H_2) \cap V(H_2')$. Then $E(G) = E_G(X_1) \cup E_G(X_2) \cup E_G(X_3)$. Since $H_2$ is pure we have

$$\bigl|E(G)\bigr| \leqslant \bigl(2|X_1| - 2\bigr) + \bigl(2|X_2| - 3\bigr) + \bigl(2|X_3| - 3\bigr) = 2\bigl|V(G)\bigr| - 2.$$

This contradicts the fact that $G$ is a mixed circuit. $\square$

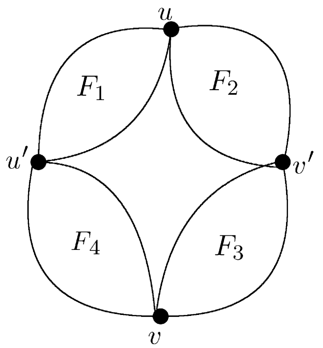


**Fig. 5.** Two crossing 2-separations $(H_1, H_2)$ and $(H_1', H_2')$ of a graph $G$: $F_1, F_2, F_3, F_4$ are subgraphs of $G$ with $F_1 \cup F_2 \cup F_3 \cup F_4 = G$ and we take $H_1 = F_1 \cup F_4$, $H_2 = F_2 \cup F_3$, $H_1' = F_1 \cup F_2$, and $H_2' = F_4 \cup F_3$.

## 4. Admissible nodes

Let $G$ be a mixed graph and $v$ be a node of $G$. The *1-reduction operation at $v$ on edges $vu, vw$* deletes $v$ and all edges incident with $v$, and adds a new edge $uw$. (This operation is called *splitting* in [1,12,14].) The type of the new edge is arbitrary, unless $v$ is a pure node, in which case the type of $uw$ must be the same as the type of $v$. The graph obtained by the operation is denoted by $G_v^{uw}$, or more simply $G_v$. We will also say that $G_v^{uw}$ *is a 1-reduction of* $G$. Note that 1-reduction is the inverse operation to 1-extension. We say that the 1-reduction $G_v^{uw}$ is a *direction 1-reduction* or a *length 1-reduction* according to the type of the new edge $uw$.

When $G$ is a mixed circuit, a 1-reduction is *admissible* if it results in a smaller mixed circuit. A node $v$ is *admissible* if $G$ has an admissible 1-reduction at $v$. Otherwise $v$ is *non-admissible*. Examples of non-admissible nodes are given in Figs. 7 and 8.

We will determine when a mixed circuit contains an admissible node. We need the following four lemmas. The first characterizes when a 1-reduction at a node $v$ is non-admissible in terms of critical sets containing two neighbours of $v$. The next three give information on the structure of families of critical sets containing pairs of neighbours $v$.

**Lemma 4.1.** *Let $G$ be a mixed circuit and let $v$ be a node in $G$ with edges $vu, vw, vt$ incident to $v$, where $u \neq w$.*

(a) *Suppose there is no admissible direction (length) 1-reduction of $G$ at $v$ on $vu, vw$. Then there exists either a mixed critical set $X$ in $G - \{v, t\}$ with $u, w \in X$, or a direction (length) critical set $Y$ in $G - v$ with $u, w \in Y$.*
(b) *Suppose that $v$ is a mixed node and there is no admissible 1-reduction of $G$ at $v$ on $vu, vw$. Then there exists either a mixed critical set $X$ in $G - \{v, t\}$ with $u, w \in X$, or there exist a direction critical set $Y$ and a length critical set $Z$ with $Y \cap Z = \{u, w\}$, $d(Y, Z) = 0$, and $Y \cup Z = V - v$.*

**Proof.** (a) Suppose there is no admissible direction 1-reduction of $G$ at $v$. Then $G - v + uw$ is not a mixed circuit, where $uw$ is a direction edge. Hence there exists either a mixed critical set $X$ in $G - v$ with $\{u, w\} \subseteq X$ and $X \neq V - v$, or a direction critical set $Y \subseteq V - v$ with $\{u, w\} \subseteq Y$. Furthermore, if the first alternative holds, then $t \notin X$ since otherwise we would have $i(X \cup v) = 2|X + v| - 1$ and $|X + v| \leqslant |V| - 1$, contradicting the fact that $G$ is a mixed circuit.

(b) Suppose that there is no mixed critical set $X$ in $G - \{v, t\}$ with $u, w \in X$. Since there is no admissible 1-reduction of $G$ at the mixed node $v$ on $vu, vw$, we may use (a) to deduce that there exist a direction critical set $Y \subseteq V - v$ with $\{u, w\} \subseteq Y$, and a length critical set $Z$ in $G - v$ with $\{u, w\} \subseteq Z$. Then $|Y \cap Z| \geqslant 2$ and we may apply Lemma 2.4(d) to $G - v$ to deduce that $Y \cup Z$ is mixed critical, $Y \cap Z = \{u, w\}$, and $d(Y, Z) = 0$. Since $G$ is a mixed circuit and $i((Y \cup Z) + v) = 2|(Y \cup Z) + v| - 1$, we have $Y \cup Z = V - v$, as required. $\square$

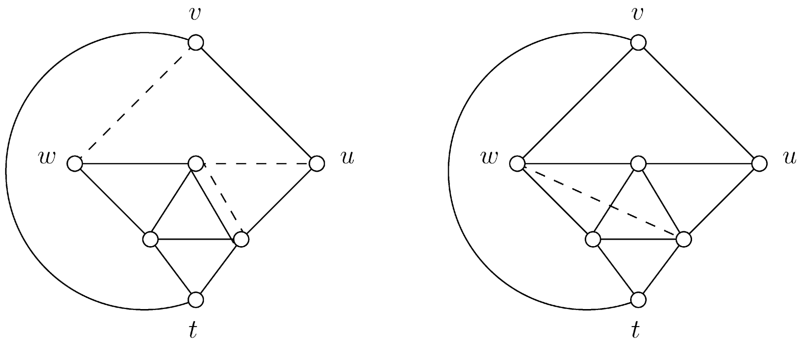


**Fig. 6.** A strong flower and a weak flower on node $v$.

For a mixed graph $G=(V;D,L)$ and $X\subseteq V$ let $N(X)$ denote the set of *neighbours* of $X$ (that is, $N(X)=\{v\in V-X\colon\ uv\in D\cup L$ for some $u\in X\}$).

**Lemma 4.2.** *Let $G=(V;D,L)$ be a mixed circuit and $v$ be a node of $G$ with three distinct neighbours $u,w$ and $t$. Suppose there exist mixed critical sets $X,Y$ in $G-v$ with $\{u,w\}\subseteq X\subseteq V-\{v,t\}$ and $\{w,t\}\subseteq Y\subseteq V-\{v,u\}$. Suppose further that one of the following conditions holds*:

(i) *there exists a mixed critical set $Z$ in $G-v$ with $\{u,t\}\subseteq Z\subseteq V-\{v,w\}$;*
(ii) *$v$ is a direction (length) pure node and there exists a direction (length) pure critical set $Z$ in $G-v$ with $\{u,t\}\subseteq Z\subseteq V-\{v\}$.*

*Let $W^*=(V-v)-W$ for each $W\in\{X,Y,Z\}$. Then*

(a) *$X\cup Y=X\cup Z=Y\cup Z=V-v$,*
(b) *$X\cap Y\cap Z\neq\emptyset$,*
(c) *$d(X,Y)=d(Y,Z)=d(X,Z)=0$,*
(d) *$Z\subseteq V-\{v,w\}$ and $\{X^*,Y^*,Z^*,X\cap Y\cap Z\}$ is a partition of $V-v$.*

**Proof.** Lemma 2.4(a) implies that $X\cap Y$ and $X\cup Y$ are both mixed critical sets in $G-v$ and $d(X,Y)=0$. Since $N(v)\subseteq X\cup Y$, we must have $X\cup Y=V-v$. Since $Z$ is critical, $G[Z]$ is connected by Lemma 2.3. Thus $X\cap Y\cap Z=\emptyset$ would imply $d(X,Y)\geqslant 1$, contradicting Lemma 2.4(a). Hence $X\cap Y\cap Z\neq\emptyset$. This implies $|X\cap Z|,|Y\cap Z|\geqslant 2$. Thus Lemma 2.4(a), (c) gives $X\cup Z=V-v$, $Y\cup Z=V-v$, and $d(X,Z)=d(Y,Z)=0$. If (i) holds then $Z\subseteq V-\{v,w\}$. On the other hand, if (ii) holds and $w\in Z$, then $G[Z+v]$ would be a pure circuit properly contained in $G$. Hence $Z\subseteq V-\{v,w\}$ in both cases. Now (a) and (b) imply that the remainder of (d) holds. □

A collection of three critical sets $X,Y,Z$ satisfying the hypotheses of Lemma 4.2 with condition (i) (respectively condition (ii)) is called a *strong* (*respectively weak*) *flower on node $v$*. We think of the sets $X^*,Y^*,Z^*$ as 'petals' of the flower and $X\cap Y\cap Z$ as its 'centre'. See Fig. 6 for examples of strong and weak flowers and Fig. 9 for a schematic drawing of a flower.

**Lemma 4.3.** *Let $G=(V;D,L)$ be a mixed circuit and $v$ be a pure node of $G$ with three distinct neighbours $u,w$ and $t$. Suppose that there exist a mixed critical set $X$ and pure critical sets $Y,Z$ of the same type as $v$ in $G-v$ with $\{u,w\}\subseteq X\subseteq V-\{v,t\}$, $\{w,t\}\subseteq Y\subseteq V-\{v\}$, and $\{u,t\}\subseteq Z\subseteq V-\{v\}$. Then there is an unbalanced 2-separation in $G$.*

**Proof.** First observe that if $|Y\cap Z|\geqslant 2$ then Lemma 2.4(b) implies that $G[(Y\cup Z)+v]$ contains a pure circuit, a contradiction. Thus $Y\cap Z=\{t\}$. Next suppose $|X\cap Y|\geqslant 2$. Then $X\cup Y$ is mixed critical and

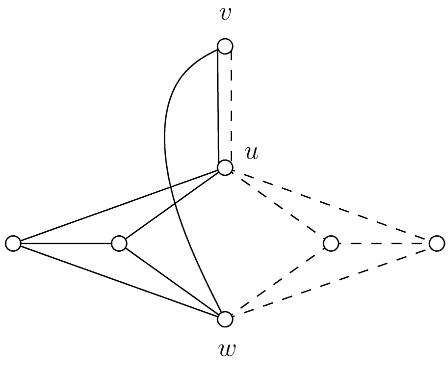


**Fig. 7.** A non-admissible mixed node $v$.

$d(X,Y)=0$ by Lemma 2.4(c). Thus $X\cup Y=V-v$. Since $Z$ is critical, $G[Z]$ is connected by Lemma 2.3. This and the fact that $(Y\cap Z)=\{t\}$ imply that $d(X,Y)\geqslant 1$, a contradiction.

Thus $|X\cap Y|=|X\cap Z|=|Y\cap Z|=1$ and $X\cap Y\cap Z=\emptyset$. Lemma 2.2 now gives that $X\cup Y\cup Z$ is mixed critical and $d(X,Y,Z)=0$. Since $N(v)\subseteq(X\cup Y\cup Z)$, we must have $X\cup Y\cup Z=V-v$. If $|Y|\geqslant 3$ then $(G[Y],G[X\cup Z\cup\{v\}])$ is an unbalanced 2-separation of $G$. Thus we may suppose that $|Y|=2$. We obtain an unbalanced 2-separation in a similar way when $|Z|\geqslant 3$. Thus we may assume that $|Y|=|Z|=2$. Since $G$ is a circuit, this implies that $|X|\geqslant 3$. Hence $(G[X],G[Y\cup Z\cup\{v\}])$ is an unbalanced 2-separation in $G$. $\square$

**Lemma 4.4.** *Let $G=(V;D,L)$ be a mixed circuit and $v$ be a pure node of $G$ with three distinct neighbours $u,w,t$. Then there cannot exist pure critical sets $X,Y,Z$ of the same type as $v$ in $G-v$ with $\{u,w\}\subseteq X\subseteq V-\{v\}$, $\{w,t\}\subseteq Y\subseteq V-\{v\}$, and $\{u,t\}\subseteq Z\subseteq V-\{v\}$.*

**Proof.** Suppose that the three sets in the lemma do exist. If $|X\cup Y|\geqslant 2$, say, then Lemma 2.4(b) implies that $G[(X\cup Y)+v]$ contains a pure circuit, a contradiction. So $|X\cap Y|=|X\cap Z|=|Y\cap Z|=1$ and $X\cap Y\cap Z=\emptyset$. Lemma 2.5(b) now gives that $(X\cup Y\cup Z)\cup\{v\}$ contains a spanning pure circuit, a contradiction. $\square$

**Lemma 4.5.** *Let $G=(V;D,L)$ be a mixed circuit and $v$ be a mixed node of $G$. Then exactly one of the following alternatives hold*:

(a) *$v$ is admissible*;
(b) *$v$ has exactly two neighbours $u,w$ and there exist a length critical set $X$ and a direction critical set $Y$ with $X\cap Y=\{u,w\}$, $X\cup Y=V-v$, and $d(X,Y)=0$*;
(c) *There is a strong flower on $v$ in $G$.*

**Proof.** Assume $v$ is not admissible. If $v$ has only two neighbours then (b) holds by Lemma 4.1. Hence we may suppose that $v$ has three distinct neighbours $u,w,t$.

We apply Lemma 4.1(b) to each pair of neighbours of $v$. Suppose the second alternative in Lemma 4.1(b) holds for some pair of neighbours, say $u,w$. Then there exist a length critical set $X$ and a direction critical set $Y$ in $G-v$ with $X\cap Y=\{u,w\}$ and $X\cup Y=V-v$. By symmetry we may suppose that $t\in X-Y$. Then all edges incident to $t$ (except possibly $vt$) are length edges, so $t$ cannot belong to a direction critical set in $G-v$. Since $v$ is not admissible, we must have a mixed critical set $Z$ in $G-v$ with $\{u,t\}\subseteq Z\subseteq V-\{v,w\}$ by Lemma 4.1. Since $u$ is a cutvertex of $G-v-w$, $Z\cap X$ is mixed critical by Lemma 2.3(b). But $E_D(Z\cap X)=\emptyset$, a contradiction.

Thus, the first alternative of Lemma 4.1(b) holds for all pairs of neighbours of $v$. Hence there exist mixed critical sets $X,Y,Z$ in $G-v$ with $\{u,w\}\subseteq X\subseteq V-\{v,t\}$, $\{w,t\}\subseteq Y\subseteq V-\{v,u\}$, and $\{u,t\}\subseteq Z\subseteq V-\{v,w\}$. Lemma 4.2(i) now implies that there is a strong flower on $v$ in $G$. $\square$

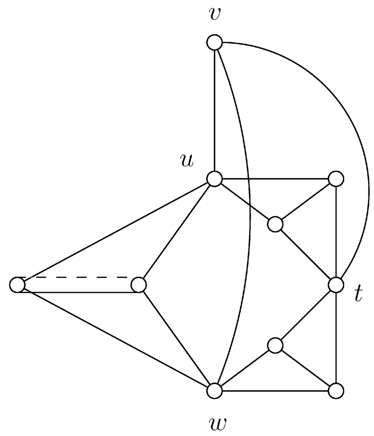


**Fig. 8.** A non-admissible pure node $v$.

Lemma 4.5 implies the following.

**Lemma 4.6.** *Let $G = (V; D, L)$ be a mixed circuit with $|V| \geqslant 4$ and let $v$ be a mixed node of $G$ with $|N(v)| = 2$. If $v$ is non-admissible then there is an unbalanced 2-separation in $G$.*

We next consider the case when $v$ is a pure node.

**Lemma 4.7.** *Let $G$ be a mixed circuit and $v$ be a pure node of $G$. If $v$ is non-admissible then either there is an unbalanced 2-separation in $G$, or there is a weak or strong flower on $v$ in $G$.*

**Proof.** We may suppose that $v$ is non-admissible and, by symmetry, that $v$ is length pure. Since $v$ is pure, we must have $|N(v)| = 3$. Since $v$ is non-admissible, Lemma 4.1(a) implies that there is a mixed critical or length critical set in $G - v$ containing each pair of neighbours of $v$. The lemma now follows from Lemmas 4.2, 4.3, and 4.4. □

Suppose $v$ is a node in a mixed circuit $G$ with $N(v) = \{x, y, z\}$. If $X$ is a critical set in $G - v$ with $x, y \in X$ and $v, z \notin X$, then we call $X$ a *$v$-critical set on $x$ and $y$*, or simply a *$v$-critical set*. If $d(z) = 3$ then the 1-reduction $G_v^{xy}$ is non-admissible, since it would decrease the degree of $z$ to two. In this case $V - \{v, z\}$ is a "trivial" $v$-critical set on $x$ and $y$. "Non-trivial" critical sets will be of particular interest: if $X$ is a $v$-critical set on $x$ and $y$ for some node $v$ with $N(v) = \{x, y, z\}$, and $d(z) \geqslant 4$, then $X$ is said to be *$v$-node-critical* or simply *node-critical*.

**Lemma 4.8.** *Let $G = (V; D, L)$ be a balanced mixed circuit and let $v \in V$ be a node. Let $N(v) = \{x, y, z\}$ with $d(z) \geqslant 4$, and let $X$ be a mixed $v$-node-critical set on $x$, $y$. Suppose that either*

(i) *there is a non-admissible series node $u$ of $G$ in $V - X - v$ with exactly one neighbour $w$ in $X$, and $w$ is a node of $G$, or*
(ii) *there is a non-admissible leaf node $t$ of $G$ in $V - X - v$.*

*Then there is a mixed node-critical set $X^*$ with $|X^*| > |X|$.*

**Proof.** Suppose that condition (i) holds. Let $N(u) = \{w, p, q\}$. By our assumption $N(u) \cap X = \{w\}$ and $d(w) = 3$. Since $u$ is a series node, we may assume that $d(p) = 3$ and $d(q) \geqslant 4$. Lemma 4.1 and the non-admissibility of $u$ imply that there exists a (pure or mixed) $u$-critical set $Y$ on $w$ and $p$. Since $G[V_3]$ is a forest by Lemma 3.4, we have $pw \notin D \cup L$ and hence $|Y| \geqslant 3$. Thus $G[Y]$ has minimum

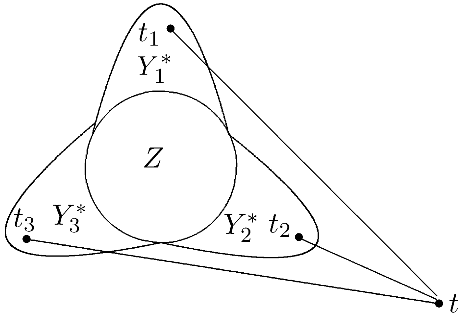


**Fig. 9.** The flower $(Y_1, Y_2, Y_3)$ on the node $t$. We have a partition $\{Y_1^*, Y_2^*, Y_3^*, Z\}$ of $V - t$, with $Y_i = (V - t) - Y_i^*$ for $1 \leqslant i \leqslant 3$.

degree at least two by Lemma 2.3(a) and hence $Y$ contains each of the two neighbours of $w$ distinct from $u$. Since $G[X]$ is connected, at least one of these neighbours of $w$ must belong to $X$. Thus $|X \cap Y| \geqslant 2$. By Lemma 2.4(a) and (c), $X^* = X \cup Y$ is a mixed $u$-critical set on $w$ and $p$. Since $d(q) \geqslant 4$ and $q \notin X$, $X^*$ is a mixed $u$-node-critical set which properly contains $X$.

Thus we may assume that condition (ii) holds. We must have $|N(t) \cap X| \leqslant 2$, since $|N(t) \cap X| = 3$ would imply that $G[X + t]$ contains a circuit and contradicts the fact that $G$ itself is a circuit. If $|N(t) \cap X| = 2$ then $X + t$ is also mixed $v$-node-critical and the lemma follows by choosing $X^* = X + t$. Thus we may assume that

$$\left|N(t) \cap X\right| \leqslant 1. \tag{4}$$

Since $G$ is balanced and $t$ is non-admissible, Lemmas 4.2, 4.5, 4.6 and 4.7 imply that $t$ has three distinct neighbours $\{t_1, t_2, t_3\}$, and there exists a flower $(Y_1, Y_2, Y_3)$ on $t$ in $G$ with $Y_i \subseteq V - \{t, t_i\}$ for $1 \leqslant i \leqslant 3$, see Fig. 9. Since $t$ is a leaf node, we may assume that neither $t_1$ nor $t_2$ are nodes. Thus $Y_1$ and $Y_2$ are two $t$-node critical sets with $Y_1 \cup Y_2 = V - t$ and at least one of $Y_1, Y_2$ is mixed critical.

Suppose that $|X| = 2$. Since $\emptyset \neq Y_1 \cap Y_2 \cap Y_3 \subseteq Y_1$ and $\{t_2, t_3\} \subseteq Y_1 - (Y_1 \cap Y_2 \cap Y_3)$ we have $|Y_1| \geqslant 3$. Similarly $|Y_2| \geqslant 3$. Since at least one of $Y_1, Y_2$ is mixed node-critical, we may take $X^* = Y_i$ for some $i \in \{1, 2\}$.

Thus we may assume that $|X| \geqslant 3$. Since $Y_1 \cup Y_2 = V - t$, $t \notin X$, and $|X| \geqslant 3$, we have $|X \cap Y_1| \geqslant 2$ or $|X \cap Y_2| \geqslant 2$. Let us assume, without loss of generality, that $|X \cap Y_1| \geqslant 2$ holds. By Lemma 2.4, $X \cup Y_1$ is mixed critical. If $t_1 \notin X$ then the lemma follows by choosing $X^* = X \cup Y_1$, which is a mixed $t$-node-critical set which properly contains $X$.

Thus we may assume that $t_1 \in X$, and hence $t_2, t_3 \notin X$ by (4). If $|X \cap Y_2| \geqslant 2$ then we are done, as above, by choosing $X^* = X \cup Y_2$. Thus we may suppose that $X \cap Y_2 = \{t_1\}$. Since $Y_1 \cup Y_2 = V - t$ and $N(t) \cap Y_1 \cap X = \emptyset$, this implies $|X| < |Y_1|$. The lemma now follows by choosing $X^* = Y_1$ if $Y_1$ is mixed.

Thus we may assume that $Y_1$ is pure. The definition of a flower now implies that $Y_2$ is mixed. Since $X \cap Y_2 \neq \emptyset$, we obtain that $X \cup Y_2$ is mixed critical by Lemma 2.4(a). Thus $X^* = X \cup Y_2$ is a mixed $t$-node-critical set which properly contains $X$, as required. □

**Theorem 4.9.** *Let $G = (V; D, L)$ be a balanced mixed circuit with $|V| \geqslant 4$. Then $G$ has an admissible node.*

**Proof.** For a contradiction suppose that $G$ is a balanced mixed circuit without admissible nodes. Since $G$ is a circuit, it has at least two nodes. Hence, by Lemma 3.4, the node subgraph of $G$ is a non-empty forest. Let $v$ be a leaf node of $G$. Since $|V| \geqslant 4$ and $G$ is balanced, it follows from Lemmas 4.5, 4.6, and 4.7 that there exists a flower on $v$. Hence there exists a mixed $v$-node-critical set $X_v$. Choose a maximum size mixed node-critical set $X_w$ with respect to some node $w$. Since $X_w + w$ is mixed critical, Lemma 3.5 implies that there is a node of $G$ in $V - w - X_w$. Let $F = G[V_3 - X]$. Since $F$ is a forest by Lemma 3.4, it contains a vertex $z$ of degree at most one. Then $z$ is a node of $G$ and $z$ is adjacent to at most one vertex of $X_w$ by the maximality of $X_w$. Thus $z$ satisfies hypothesis (i) or (ii)

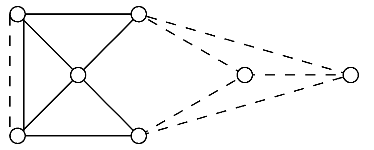

**Fig. 10.** A direction balanced mixed circuit with no admissible nodes.

of Lemma 4.8 and we may deduce that there is a mixed node-critical set $X^*$ with $|X^*| > |X_w|$. This contradicts the choice of $X_w$ and completes the proof. □

The mixed circuit in Fig. 10 shows that the hypothesis of Theorem 4.9 that $G$ is balanced cannot be weakened to direction (or length) balanced.

We may strengthen Theorem 4.9 by using the following result on the existence of admissible nodes in pure circuits (where a node in a pure circuit $G$ is *admissible* if some 1-reduction of $G$ at $v$ results in a smaller pure circuit).

**Theorem 4.10.** *(See [1].) Let $G$ be a 3-connected pure circuit with at least five vertices and $x, y, z$ be vertices of $G$ with $xy$ an edge of $G$. Then $G$ has an admissible node distinct from $x, y, z$.*

**Theorem 4.11.** *Let $G = (V; D, L)$ be a mixed circuit with $|V| \geqslant 4$. Then either $G$ can be expressed as a 2-sum of a mixed circuit with a pure $K_4$, or $G$ has an admissible node.*

**Proof.** Suppose that $G$ has no admissible nodes. By Theorem 4.9 this implies that there is a 2-separation $(H_1, H_2)$ in $G$ for which $H_2$ is pure. Choose the 2-separation so that $H_2$ is minimal. Let $V(H_1) \cap V(H_2) = \{a, b\}$. Let $H'$ be obtained from $H_2$ by adding an edge $ab$ whose type is the same as that of $H_2$. Then $H'$ is 3-connected by the minimality of $H_2$ and is a pure circuit by Lemma 3.7(b). If $|V(H')| = 4$ then $H'$ is isomorphic to $K_4$ and the theorem follows. Thus we may assume that $|V(H')| \geqslant 5$ holds. Then Theorem 4.10 implies that $H'$ has an admissible node $v$, different from $a, b$. Let $H'_v$ be obtained from $H'$ by an admissible 1-reduction at $v$. Since the 2-sum of $H_1$ and $H'_v$ is a mixed circuit by Lemma 3.7(a), it follows that $v$ is admissible in $G$, a contradiction. This completes the proof. □

Theorem 4.11 and Lemmas 3.6 and 3.7 lead to the following inductive construction for mixed circuits, and hence solve an open problem raised by Servatius and Whiteley in [18].

**Theorem 4.12.** *Let $G$ be a mixed graph. Then $G$ is a mixed circuit if and only if $G$ can be obtained from $K_3^+$ or $K_3^-$ by a sequence of 1-extensions and 2-sums with pure $K_4$'s.*

We close this section with one more lemma on admissible mixed nodes, which we will need for our characterization of globally rigid mixed circuits.

**Lemma 4.13.** *Let $G = (V; D, L)$ be a direction balanced mixed circuit and $v$ be a mixed node of $G$. Suppose that $G_v^{xy}$ is an admissible length 1-reduction at $v$. Suppose further that $G_v^{xy}$ contains a 2-separation $(H_1, H_2)$ in which $H_2$ is length pure and $xy \in E(H_2)$. Then there is an admissible direction 1-reduction at $v$.*

**Proof.** Let $V(H_1) \cap V(H_2) = \{u, v\}$. Since the length 1-reduction on $xy$ is admissible, $G_v^{x,y}$ is a mixed circuit. Using Lemma 3.7(b), we deduce that $G = G_1 \oplus_2 G_2$, where $G_1 = H_1 + uv$ is a mixed circuit, $G_2 = H_2 + uv$ is a length circuit, and $uv$ is a length edge. This implies that $uv$ is not a length edge of $G_v^{x,y}$ and hence $\{x, y\} \neq \{u, v\}$. Since $xy \in E(H_1)$, at least one of $x$ and $y$, say $x$, is in $H_1 - H_2$. Thus $x$ is length pure in $G - v$, and hence cannot belong to a direction critical set in $G - v$. Since the length 1-reduction on $xy$ is admissible, the pair $x, y$ cannot belong to a mixed critical set in $G - v$ either. Lemma 4.1(a) now implies that the direction 1-reduction on $xy$ is admissible. □

## 5. Feasible nodes

We saw in Lemma 1.6(c) that globally rigid generic mixed frameworks are direction balanced. We shall show in the next section that this necessary condition for global rigidity is also sufficient when the underlying graph is a mixed circuit. Our proof uses induction on the size of the circuit and relies on the recursive construction for direction balanced mixed circuits which we will derive in this section.

A 1-reduction in a direction-balanced mixed circuit $G$ is *feasible* if it results in a smaller direction-balanced mixed circuit. We say that a node $v$ of $G$ is *feasible* if it has a feasible 1-reduction, and otherwise that $v$ is *infeasible*.

**Lemma 5.1.** *Let $v$ be an admissible node of a direction-balanced mixed circuit $G$ and $G_v$ be the mixed circuit obtained by performing an admissible 1-reduction at $v$. Suppose that $G_v$ is not direction balanced. Then $G_v$ has a 2-separation $(H_1, H_2)$ such that $H_2$ is length pure. Furthermore, for every such 2-separation of $G_v$, $H_2 - H_1$ contains a neighbour of $v$, and, if $v$ is length pure, then $H_1 - H_2$ also contains a neighbour of $v$.*

**Proof.** Since $G_v$ is not direction balanced, $G_v$ has a 2-separation $(H_1, H_2)$ where $H_2$ is length pure. Since $G$ is direction-balanced, $(H_1 + v, H_2)$ is not a 2-separation of $G$ and hence $H_2 - H_1$ contains a neighbour of $v$. If $v$ is length pure then the fact that $G$ is direction-balanced also implies that $(H_1, H_2 + v)$ is not a 2-separation of $G$ and hence $H_1 - H_2$ contains a neighbour of $v$. □

**Theorem 5.2.** *Suppose $G$ is a direction-balanced mixed circuit with at least four vertices. Then either $G$ can be expressed as a 2-sum of a direction-balanced circuit and a direction pure $K_4$, or $G$ has a feasible node.*

**Proof.** We proceed by contradiction. Suppose the theorem is false and let $G$ be a counterexample.

Suppose that $G = G_1 \oplus_2 G_2$ for some mixed circuit, $G_1$, and pure $K_4$, $G_2$. Since $G$ is direction-balanced, $G_2$ must be direction-pure. The fact that $G$ is direction balanced now implies that $G_1$ is direction-balanced. This contradicts the fact that $G$ is a counterexample. Thus $G$ cannot be expressed as a 2-sum of a mixed circuit and a pure $K_4$, and hence $G$ has an admissible node by Theorem 4.11.

We say that an admissible 1-reduction of $G$ at a node $v$ is *acceptable* if it is an admissible direction 1-reduction at $v$ if such a 1-reduction exists (and so is an admissible length 1-reduction only when no admissible direction 1-reduction at $v$ exists). Choose an acceptable 1-reduction $G_w^{x,y}$ of $G$. Since $G$ has no feasible nodes, there exists a 2-separation $(H_1, H_2)$ of $G_w^{x,y}$ such that $H_2$ is length pure. We may suppose that $w$ and $(H_1, H_2)$ have been chosen such that $H_2$ has as few vertices as possible. Let $V(H_1) \cap V(H_2) = \{u, v\}$ and $N_G(w) = \{x, y, z\}$. Let $G_i$ be obtained by adding a length edge $uv$ to $H_i$ for each $i \in \{1, 2\}$. Using Lemma 3.7(b) and the minimality of $H_2$ we have:

**Claim 5.3.** *$G_1$ is a mixed circuit, $G_2$ is a 3-connected length-pure circuit, $G_w^{x,y}$ is a 2-sum of $G_1$ and $G_2$ along the length edge $uv$, and hence $uv$ is not a length edge of $G_w^{x,y}$.*

We shall prove that $H_2$ contains a feasible node of $G$.

**Claim 5.4.** *Either $\{x, y, z\} \cap V(H_1 - H_2) \neq \emptyset$ or $\{x, y\} = \{u, v\}$ and $G_w^{x,y}$ is a direction 1-reduction of $w$ onto $xy$.*

**Proof.** Suppose the claim is false. Then $\{x, y, z\} \subseteq V(H_2)$ and $\{x, y\} \neq \{u, v\}$. (Note that, if $\{x, y\} = \{u, v\}$, then $G_w^{x,y}$ cannot be a length 1-reduction of $w$ since Claim 5.3 tells us that $uv$ is not a length edge of $G_w^{x,y}$.) Since $G$ is direction balanced, and $H_2$ is length pure, $w$ must be a mixed node of $G$ and $xy$ must be a length edge of $G_w^{x,y}$. Lemma 4.13 now implies that $G$ has an admissible direction 1-reduction at $w$. This contradicts the fact that $G_w^{x,y}$ is an acceptable 1-reduction of $G$. □

**Claim 5.5.** *No node of $G_2$ in $V(G_2) - \{u, v, x, y, z\}$ is admissible.*

**Proof.** Suppose $b \in V(G_2) - \{u, v, x, y, z\}$ is an admissible node of the length-pure circuit $G_2$. Let $(G_2)_b^{c,d}$ be an admissible 1-reduction of $G_2$. Then $(G_2)_b^{c,d}$ is a length-pure circuit. By Lemma 3.7(a),

$$H = G_1 \oplus_2 (G_2)_b^{c,d} = \left(G_w^{x,y}\right)_b^{c,d}$$

is a mixed circuit. Since $G_b^{c,d}$ is a 1-extension of $H$, Lemma 3.6 implies that $G_b^{c,d}$ is a mixed circuit and hence $G_b^{c,d}$ is an admissible 1-reduction in $G$. Since $b$ is a length-pure node of $G$, $G_b^{c,d}$ is acceptable. Lemma 5.1 implies that $G_b^{c,d}$ has a 2-separation $(H_1', H_2')$ where $H_2'$ is length-pure and both $H_1' - H_2'$ and $H_2' - H_1'$ contain a neighbour of $b$. Let $V(H_1') \cap V(H_2') = \{u', v'\}$. Since $u', v'$ have degree at least four in $G_b^{c,d}$ by Lemma 3.7(b), they have degree at least four in $G$. Thus $w \notin \{u', v'\}$.

Since $\{u', v'\}$ is a 2-vertex-cut of $G_b^{c,d}$, it is also a 2-vertex-cut of $H$. Similarly $\{u, v\}$ is a 2-vertex-cut of $H$. Since $(G_2)_b^{c,d}$ is a circuit, it is 2-connected by Lemma 3.3. Thus $(G_2)_b^{c,d} - u$ and $(G_2)_b^{c,d} - v$ are both connected. Since $N_G(b) \subseteq V(G_2)$, and since $\{u', v'\}$ separates two of the neighbours of $b$ in $G_b^{c,d}$, we must have either $\{u', v'\} = \{u, v\}$ or $\{u', v'\} \cap (V(G_2) - \{u, v\}) \neq \emptyset$. Applying Lemma 3.8 to $H$ if the latter alternative holds, we have $\{u', v'\} \subseteq V(G_2)$ in both cases. Thus $V(H_1) \subseteq V(H_1')$. If the first alternative of Claim 5.4 holds, then $w$ is adjacent to at least one vertex of $H_1 - H_2$. If the second alternative of Claim 5.4 holds, then $w$ is not a length pure node of $G$. We may deduce in both cases that $w$ is contained in $H_1'$. Thus $V(H_1) \cup \{w\} \subseteq V(H_1')$. Since $|V(H_1)| + |V(H_2)| = |V(H_1')| + |V(H_2')|$, this implies that $|V(H_2')| < |V(H_2)|$, and contradicts the minimality of $H_2$. □

**Claim 5.6.** *$G_2$ is isomorphic to $K_4$.*

**Proof.** Suppose $G_2$ is not isomorphic to $K_4$. If $\{x, y\} \subseteq V(H_1)$ then we may use Theorem 4.10 and the fact that $uv \in E(G_1)$ to contradict Claim 5.5. Thus $\{x, y\} \nsubseteq V(H_1)$. Since $xy \in E(G_w^{x,y})$ and $(H_1, H_2)$ is a 2-separation of $G_w^{x,y}$, this implies that $\{x, y\} \subseteq V(H_2)$. Claim 5.4 now gives $z \in V(H_1 - H_2)$. If $\{x, y\} \cap \{u, v\} \neq \emptyset$ then we may again use Theorem 4.10 to contradict Claim 5.5. Hence $\{x, y\} \subseteq V(H_2 - H_1)$. Theorem 4.10 and Claim 5.5 now imply that $u, v, x, y$ are the only admissible nodes in $G_2$. Since $G_w^{x,y}$ is an acceptable 1-reduction of $G$, Lemma 4.13 implies that $w$ is a length-pure node of $G$. We shall show that $x$ is a feasible node in $G$.

Since $x$ is an admissible node of $G_2$, $(G_2)_x^{s,t}$ is a pure circuit for some $s, t \in N_{G_2}(x)$. Let $N_{G_2}(x) = \{q, s, t\}$. Since $xy$ is an edge of $G_2$ and $y$ is a node of $G_2$, we must have $y \in \{s, t\}$. Without loss of generality, $y = t$. By Lemma 3.7(a), $H = (G_w^{x,y})_x^{s,y} = G_1 \oplus_2 (G_2)_x^{s,y}$, is a mixed circuit. Since $G_x^{s,w}$ is a 1-extension of $H$, Lemma 3.6 implies that $G_x^{s,w}$ is a mixed circuit. Thus $x$ is an admissible node of $G$. Since $x$ is length-pure, $G_x^{s,w}$ is an acceptable 1-reduction of $G$. Lemma 5.1 now implies that $G_x^{s,w}$ has a 2-separation $(H_1', H_2')$ where $H_2'$ is length-pure and $H_1' - H_2'$ and $H_2' - H_1'$ both contain a neighbour of $x$ in $G$. Let $V(H_1') \cap V(H_2') = \{u', v'\}$. Since $u', v'$ have degree at least four in $G_x^{s,w}$ by Lemma 3.7(b), they have degree at least four in $G$. Thus $w \notin \{u', v'\}$.

We proceed as in the proof of Claim 5.5. Since $G$ is direction-balanced, $\{u', v'\}$ separates $w$ and $q$ in $G_x^{s,w}$. Since $\{u', v'\}$ is a 2-vertex-cut of $G_x^{s,w}$, it is also a 2-vertex-cut of $H$. Similarly $\{u, v\}$ is a 2-vertex-cut of $H$. Since $(G_2)_x^{s,y}$ is a circuit, it is 2-connected. Thus the graph $F$ obtained from $(G_2)_x^{s,y} - sy$ by adding the vertex $w$ and edges $sw, yw$ is 2-connected. Since $N_G(x) \subseteq V(F)$ and $F - uv \subseteq G_x^{s,y}$, and since $\{u', v'\}$ separates two of the neighbours of $x$ in $G_x^{s,w}$, we must have either $\{u', v'\} = \{u, v\}$ or $\{u', v'\} \cap (V(G_2) - \{u, v\}) \neq \emptyset$. Applying Lemma 3.8 to $H$ if the latter alternative holds, we have $\{u', v'\} \subseteq V(G_2)$ in both cases. Thus $V(H_1) \subseteq V(H_1')$. Furthermore, since $z \in V(H_1 - H_2)$ and $wz \in E(G_x^{s,w})$, we have $w \in V(H_1')$. Thus $|V(H_2)| > |V(H_2')|$. This contradicts the minimality of $H_2$. □

**Claim 5.7.** $\{x, y\} \neq \{u, v\}$.

**Proof.** Suppose $\{x, y\} = \{u, v\}$. Since $G$ is direction balanced, $z \in V(H_2) - \{x, y\}$ and $w$ is not a length pure node of $G$. Claim 5.4 now implies that $G_w^{x,y}$ is a direction 1-reduction of $w$ onto $xy$ and hence $xy$ is a direction edge of $H_1$. Let $V(H_2) = \{x, y, z, t\}$. Then $t$ is a length-pure node of $G$.

Let $G_t^{x,y}$ be obtained by performing a 1-reduction at $t$ onto a length edge $xy$. Then $G_t^{x,y}$ can be constructed from $G_1$ by two 1-extensions. (We first delete the direction edge $xy$, add the vertex $z$, length edges $zx, zy$ and a direction edge $zx$. We then delete the direction edge $zx$, add $w$ and edges $wx, wy, wz$ of the same type as in $G$.) Thus $G_t^{x,y}$ is a mixed circuit by Lemma 3.6. Lemma 5.1 now implies that $G_t^{x,y}$ has a 2-separation $(H_1', H_2')$ where $H_2'$ is length-pure and both $H_1' - H_2'$ and $H_2' - H_1'$ contain a neighbour of $t$. This is impossible since the neighbours of $t$ in $G$ induce a complete graph in $G_t^{x,y}$. Thus $\{x, y\} \neq \{u, v\}$. □

**Claim 5.8.** *$\{x, y\} \subset V(H_2)$, $w$ is a length-pure node of $G$ and $z$ is a vertex of $H_1 - H_2$.*

**Proof.** Suppose that $\{x, y\} \not\subset V(H_2)$. Since $xy$ is an edge of $G_w^{x,y}$, we must have $\{x, y\} \subseteq V(H_1)$ and $z \in V(H_2 - H_1)$. Choose $t \in V(H_2) - \{u, v, z\}$. We have $V(H_2) = \{u, v, z, t\}$ and $t$ is a length-pure node of $G$. Let $G_t^{u,v}$ be obtained by performing a 1-reduction of $t$ onto a length edge $uv$. Then $G_t^{u,v}$ can be constructed from $G_1$ by two 1-extensions. (We first delete the edge $xy$ and add the vertex $w$ and edges $wx, wy, wv$, where $wx, wy$ are of the same type as in $G$ and $wv$ is of the same type as the edge $wz$ in $G$. This is an 'allowed' 1-extension since, if $w$ is pure in $G$, then the edge $xy$ constructed by the 1-reduction $G_w^{x,y}$ has the same type as $w$. We then delete the edge $wv$ and add vertex $z$ and edges $zu, zv, zw$ of the same type as in $G$.) Hence $G_t^{u,v}$ is a mixed circuit by Lemma 3.6. Lemma 5.1 now implies that $G_t^{u,v}$ has a 2-separation $(H_1', H_2')$ where $H_2'$ is length-pure and both $H_1' - H_2'$ and $H_2' - H_1'$ contain a neighbour of $t$. This is impossible since the neighbours of $t$ in $G$ induce a complete graph in $G_t^{u,v}$. Thus $\{x, y\} \subset V(H_2)$.

We may now use Lemma 4.13, Claim 5.7 and the fact that $G_w^{x,y}$ is an acceptable 1-reduction of $G$ to deduce that $w$ is length-pure. Since $G$ is direction balanced, this implies that $z$ is in $H_1 - H_2$. □

**Claim 5.9.** *$\{x, y\} \cap \{u, v\} \neq \emptyset$.*

**Proof.** Suppose the claim is false. Then $x$ and $y$ are both length-pure nodes of $G$. Let $G_x^{w,v}$ be obtained by performing a 1-reduction of $G$ at $x$ onto a length edge $wv$. Note that $wv \notin E(G)$ since $z$ is in $H_1 - H_2$. Note further that $G_x^{w,v}$ can be obtained from $G_1$ by a sequence of two 1-extensions. Thus $G_x^{w,v}$ is a mixed circuit by Lemma 3.6. Lemma 5.1 implies that $G_x^{w,v}$ has a 2-separation $(H_1', H_2')$ where $H_2'$ is length-pure and both $H_1' - H_2'$ and $H_2' - H_1'$ contain neighbours of $x$. Since each of the neighbours of $x$ in $G$ is a neighbour of $y$ in $G_x^{w,v}$, we must have $y \in V(H_1') \cap V(H_2')$. This contradicts Lemma 3.7(b) since $y$ has degree three in $G_x^{w,v}$. □

We can now complete the proof of the theorem. Using Claims 5.6, 5.7, 5.8 and 5.9, and relabelling if necessary, we may suppose that $y = v$ and $V(H_2) = \{u, y, x, t\}$. Thus $x$ and $t$ are length-pure nodes of $G$. Let $G_x^{w,t}$ be obtained by performing a 1-reduction of $G$ at $x$ onto a length edge $wt$. Note that $wt \notin E(G)$ since $z$ is in $H_1 - H_2$. Note further that $G_x^{w,t}$ can be obtained from $G_1$ by a sequence of two 1-extensions. Thus $G_x^{w,t}$ is a mixed circuit by Lemma 3.6. Lemma 5.1 implies that $G_x^{w,t}$ has a 2-separation $(H_1', H_2')$ where $H_2'$ is length-pure and both $H_1' - H_2'$ and $H_2' - H_1'$ contain neighbours of $x$. Since both the neighbours of $x$ in $G - t$ are neighbours of $t$ in $G_x^{w,t}$, we must have $t \in V(H_1') \cap V(H_2')$. This contradicts Lemma 3.7(b) since $t$ has degree three in $G_x^{w,t}$. □

**Theorem 5.10.** *Let $G = (V; D, L)$ be a mixed graph. Then $G$ is a direction-balanced mixed circuit if and only if $G$ can be obtained from $K_3^+$ or $K_3^-$ by 1-extensions and 2-sums with direction-pure $K_4$'s.*

**Proof.** It is easy to see that the operations of 1-extension and taking a 2-sum with a direction-pure $K_4$ preserve the property of being a direction-balanced circuit. We may verify the reverse implication by induction on $|V|$ using Theorem 5.2. □

## 6. Globally rigid circuits

We can now obtain our promised characterization of globally rigid mixed circuits. We need one final lemma.

**Lemma 6.1.** *Suppose $G$ is a mixed graph and $G = G_1 \oplus_2 G_2$ where $G_1$ is globally rigid and $G_2$ is a direction-pure $K_4$. Then $G$ is globally rigid.*

**Proof.** It is straightforward to check that $G$ can be constructed from $G_1$ by a direction 0-extension and a direction 1-extension. The lemma follows since these operations preserve global rigidity by Theorems 1.7 and 1.8, respectively. $\square$

**Theorem 6.2.** *Let $(G, p)$ be a generic realization of a mixed circuit. Then $(G, p)$ is globally rigid if and only if $G$ is direction-balanced.*

**Proof.** Necessity follows from Lemma 1.6(c). Sufficiency follows from Theorem 5.10 using the facts that both the mixed circuits with three vertices are globally rigid, and that the operations of 1-extension and 2-sum with a direction-pure $K_4$ preserve global rigidity by Theorem 1.7 and Lemma 6.1, respectively. $\square$

Note that Theorem 6.2 implies that global rigidity is a generic property for mixed circuits since the global rigidity of a generic realization $(G, p)$ of a mixed circuit depends only on the mixed circuit $G$ and not the map $p$.

## 7. Concluding remarks

### 7.1. Algorithmic considerations

There exist efficient algorithms to check whether a mixed graph $G = (V; D, L)$ satisfies sparsity conditions (1) and (2) of Theorem 1.5. Condition (1) holds if and only if the edge set of the unlabeled graph $H = (V, D \cup L)$ can be covered by two forests, which can be tested in $O(n^{3/2} \log(n^2/m))$ time [6], where $n$ and $m$ denote the number of vertices and edges, respectively. Condition (2) is equivalent to independence in the well-known length rigidity matroid and can be tested in $O(n^2)$ time, see [2] and the references therein. By using these algorithms one can test independence in the direction–length rigidity matroid, check whether $G$ is a mixed (or pure) circuit, and obtain the inductive construction of Theorem 4.12 in polynomial time.

Testing whether $G$ is direction balanced can be done in linear time. This follows by observing that $G$ is direction balanced if and only if all 2-separations $(H_1, H_2)$ of $G$, in which $H_2$ is minimal, are direction balanced. It is straightforward to obtain these special 2-separations from the cleavage units (or 3-connected components) of $H$, which can be listed in $O(n + m)$ time [11]. Thus one can also check whether $G$ is a direction balanced mixed circuit and obtain the inductive construction of Theorem 5.10 in polynomial time.

### 7.2. Globally linked pairs of vertices

The results of this paper, together with [13], can be used to characterize the 'globally linked pairs', the 'globally rigid clusters', and the 'uniquely localizable vertices' in an arbitrary mixed circuit. These notions were introduced for length frameworks in [14].

A pair of vertices $\{u, v\}$ in a mixed framework $(G, p)$ is *globally linked* in $(G, p)$ if, in all equivalent frameworks $(G, q)$, we have that $p(u) - p(v)$ is a scalar multiple of $q(u) - q(v)$ and $\|p(u) - p(v)\| = \|q(u) - q(v)\|$. The pair $\{u, v\}$ is *globally linked* in $G$ if it is globally linked in all generic frameworks $(G, p)$. Thus $G$ is globally mixed-rigid if and only if all pairs of vertices of $G$ are globally linked. A *globally rigid cluster* of a mixed graph $G = (V; D, L)$ is a maximal subset of $V$ in which all pairs of vertices are globally linked in $G$.

The *core* of a mixed graph $G$ is defined to be the maximal subgraph of $G$ in which no pairs of vertices are separated by a direction unbalanced 2-separation. It can be seen that the core of $G$ is unique and is equal to the subgraph $C$ of $G$ obtained by 'cleaving off' the pure sides of the direction unbalanced 2-separations of $G$. Globally linked pairs and globally rigid clusters in a mixed circuit are determined by its core:

**Theorem 7.1.** *(See [13].) Let $G=(V;D,L)$ be a mixed circuit and let $C$ be its core. Then*

(a) *a pair $\{u,v\}\subseteq V$ is globally linked in $G$ if and only if $\{u,v\}\subseteq V(C)$,*
(b) *$V(C)$ is the only globally rigid cluster of $G$.*

Note that, in the analogous result to Theorem 7.1(b) for length frameworks given in [14], a graph $G=(V,E)$ may contain several globally rigid clusters and the union of these clusters is equal to $V$.

By using similar techniques to [14] it is also possible to characterize 'uniquely localizable vertices' in a mixed circuit, with respect to a given set $P\subseteq V$ of 'pinned' vertices. We can also determine the number of non-congruent generic realizations of an arbitrary mixed circuit.

### 7.3. A family of globally rigid d-dimensional generic frameworks

**Theorem 7.2.** *Let $G=(V;D,L)$ be a mixed graph in which all pairs of adjacent vertices are connected by both a length and a direction edge, and $(G,p)$ be a $d$-dimensional generic realization of $G$. Then $(G,p)$ is globally rigid if and only if $G$ is 2-connected.*

**Proof.** Necessity follows from (the $d$-dimensional analogue of) Lemma 1.6(b). To verify sufficiency suppose that $G$ is 2-connected and let $H=(V,L)$ be the underlying length-pure spanning subgraph of $G$. Let $(G,q)$ be a realization of $G$ which is equivalent to $(G,p)$ and $u,v$ be adjacent vertices of $G$. By applying a suitable translation and dilation by $-1$ to $(G,q)$, if necessary, we may suppose that $p(u)=q(u)$ and $p(v)=q(v)$. Let $p=(p_1,p_2,\dots,p_d)$ and $q=(q_1,q_2,\dots,q_d)$ where $p_i,q_i:V\to\mathbb{R}$ for all $1\leqslant i\leqslant d$. Since $(G,p)$ and $(G,q)$ are equivalent, and all pairs of adjacent vertices are connected by both a length and a direction edge, we have $p(x)-p(y)=\pm(q(x)-q(y))$ for all adjacent $x,y\in V$. Hence $p_i(x)-p_i(y)=\pm(q_i(x)-q_i(y))$ for all adjacent $x,y\in V$, and $(H,p_i)$ and $(H,q_i)$ are length-equivalent 1-dimensional length frameworks. Since $H$ is 2-connected and $(H,p_i)$ is generic, $(H,p_i)$ is globally length-rigid in 1-dimensional space. Since $p_i(u)=q_i(u)$ and $p_i(v)=q_i(v)$, we must have $p_i(x)=q_i(x)$ for all $x\in V$. This holds for all $1\leqslant i\leqslant d$ and hence $p(x)=q(x)$ for all $x\in V$. □

Note that Theorem 7.2 implies that global rigidity is a generic property for the family of mixed graphs in which all pairs of adjacent vertices are connected by both a length and a direction edge. Note also that Theorem 7.2 holds within the larger family of pseudo-frameworks since the hypothesis that all pairs of adjacent vertices of $G$ are joined by a length edge ensures that no pseudo-framework which is equivalent to a generic realization of $G$ can be degenerate.

### 7.4. Strongly globally rigid frameworks

Let $G=(V;D,L)$ be a mixed graph and $(G,p)$, $(G,q)$ be 2-dimensional mixed frameworks. We say that $(G,p)$ and $(G,q)$ are *strongly equivalent* if edges in $L$ have the same length in $(G,p)$ and $(G,q)$, and edges in $D$ have the same 'oriented direction', i.e. $p(u)-p(v)=k(q(u)-q(v))$ for some $k>0$. The frameworks $(G,p)$ and $(G,q)$ are *strongly congruent* if $(G,p)$ can be obtained from $(G,q)$ by a translation. We can use these concepts to define *strong rigidity* and *strong global rigidity* of mixed frameworks. Clearly strong rigidity is the same as rigidity since it is a local property, but this is not true for global rigidity. The following example shows that strong global rigidity is not a generic property. Let $(H,p)$ be a strongly globally rigid generic mixed framework and let $(G,p')$ be a generic mixed framework obtained from $(H,p)$ by a 0-extension which adds a vertex $v$ incident with one length edge $vu$ and one direction edge $vw$. Then the strong global rigidity of $(G,p')$ depends on whether the length of $vu$ is smaller (not strongly globally rigid) or greater (strongly globally rigid) than the distance between $u$ and $w$, see Fig. 11.

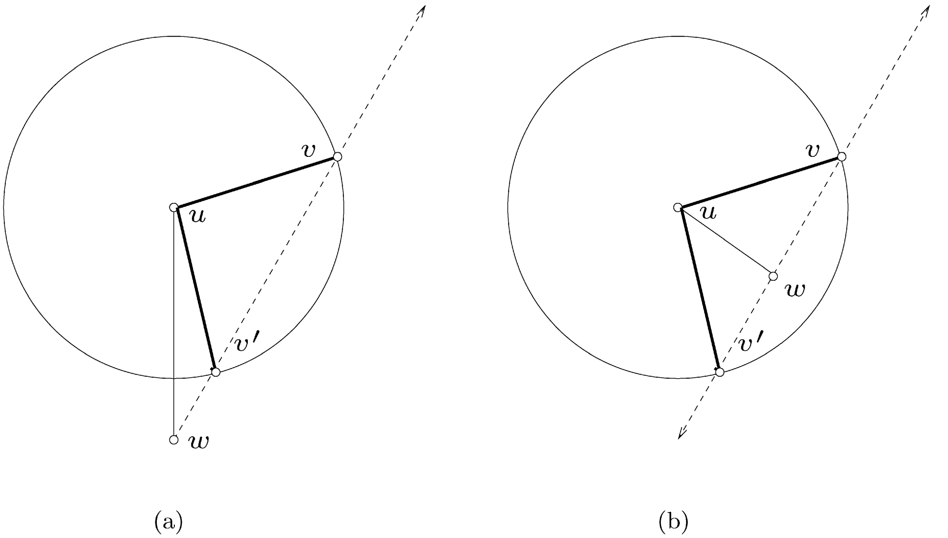


**Fig. 11.** Adding a vertex $v$ incident with one length edge $vu$ and one direction edge $vw$ can either destroy strong global rigidity, case (a), or preserve it, case (b).

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